Embedding finite lattices as initial segments of the lattice of Π^0_1 classes modulo finite differences.

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Slides available at:

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Review

Definition: A *lattice* is a partially ordered set which is pairwise closed under least upper bound and greatest lower bound, contains a maximal element 1, and a minimal element 0. We denote the least upper bound ("join") of a and b with $a \vee b$, and the greatest lower bound ("meet") with $a \wedge b$.

Definition: A lattice is *distributive* if the following hold:

$$\begin{aligned} a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c) \end{aligned}$$

Notation: We view members of $\omega^{<\omega}$ as strings. $|\tau|$ is the length of τ, i.e. the size of the domain. We say $\tau \preceq \sigma$ if $\tau \subseteq \sigma$.

Definition: $T \subseteq \omega^{<\omega}$ is a *tree* if for every $\tau \in T$ and $\sigma \preceq \tau$, $\sigma \in T$.

Notation: For a tree T, let $[T] \subseteq \omega^\omega$ denote the set of infinite paths through T.

Background

Definition: A lattice satisfies the *reduction* property if:
$\forall a, b\ \exists a' \leq a, b' \leq b\ [(a' \wedge b' = 0) \text{ and } (a' \vee b' = a \vee b)]$.

Definition: A lattice satisfies *dual-reduction* if:
$\forall a, b\ \exists a' \geq a, b' \geq b\ [(a' \wedge b' = a \wedge b) \text{ and } (a' \vee b' = 1)]$.

Definition: A lattice is a *Boolean algebra* if
$\forall a\ \exists a'\ [(a \wedge a' = 0) \text{ and } (a \vee a' = 1)]$.

Definition: A Π^0_1 class is a set $P \subseteq \omega^\omega$ of infinite paths through a computable tree, i.e. $P = [T]$.

Result 1: The computable enumerable sets form a distributive reduction lattice under inclusion, $\mathcal{E}$.

Result 2: The Π^0_1 sets form a distributive dual-reduction lattice under inclusion, $\mathcal{E}_\Pi$.

Background (cont.)

Notation: Let $\mathcal{L}(P) = [0, P]$, i.e. $\{Q \in \mathcal{E}_\Pi : Q \subseteq P\}$.

Notation: Let $\mathcal{E}^*_\Pi$ be the lattice $\mathcal{E}_\Pi$ modulo finite differences. Similarly, let $\mathcal{L}^*(P)$ be the lattice $\mathcal{L}(P)$ modulo finite differences.

Result 3: $\mathcal{L}(P)$ and $\mathcal{L}^*(P)$ satisfy dual-reduction. [1]

Result 4: If we change "computable" to "co-c.e." in the definition of Π^0_1 classes we get the same classes. As a result an effective indexing of the c.e. sets gives an effective indexing of co-c.e. trees and thus an effective indexing of Π^0_1 classes. Let P_e be the e^{th} Π^0_1 class.

Main Result

Theorem: For any finite dual-reduction distributive lattice L there exists a Π^0_1 class Q such that $\mathcal{L}^*(Q)$ is isomorphic to L. Furthermore, the theory of $\mathcal{L}(Q)$ is decidable. [1]

Corollary: $\mathcal{E} \neq \mathcal{E}_\Pi$ [1]

Proof: By Nies [4] each interval of $\mathcal{E}$ which is not a Boolean algebra has undecidable theory. Now let L be the lattice with elements $\emptyset, \{0\}, \{0,1\}$, Q as in the main theorem, and P_0 be a representative of $\{0\}$ (note $\emptyset$ and Q are representatives of $\emptyset$ and $\{0,1\}$, respectively).

Let $P \subseteq Q$. Then either $P =^* \emptyset$, i.e. finite; $P =^* Q$, i.e. cofinite; or $P =^* P_0$, i.e. cofinite in P_0 and finite in $P - P_0$. This last type has no complement and thus $\mathcal{L}(Q)$ is not a Boolean algebra. But by the main result $\mathrm{Th}(\mathcal{L}(Q))$ is decidable. $\square$

Representation of lattices.

We now work towards a proof of the main theorem.

Fix L a finite dual-reduction distributive lattice.

By a result of Herrmann [2] there exists a tree S with root $\emptyset$ which generates L in the sense that every element L is the finite join of elements of S.

We can assume that S has a single atom. For multiple atoms we construct a solution for each atom and do a tree union of the solutions:
$T_1 + T_2 = \{0^\frown\sigma : \sigma \in T_1\} \cup \{1^\frown\sigma : \sigma \in T_2\}$.

Let $m = |S| - 2$. For coding purposes we will assign each node above the root a number in $\{0, \ldots, m\}$. We also view S as a subset of $\mathcal{P}(\{0, \ldots, m\})$ where each node is the set of its number and those of its ancestors.

We define $\leq_*$ on $\{0, \ldots, m\}$ as induced by the ancestor relation.

Construction

We will build a computable tree T and define $Q = [T]$. For each $A \in L$ we will have a $Q_A \subseteq Q$ such that $A \subseteq B \Leftrightarrow Q_A \subseteq Q_B$. Finally for any $P_e \subseteq Q$ we will have $P_e =^* Q_A$ for some $A \in L$.

For $i \in \{0, \ldots, m\}$ we define the label for i to be $0^\frown 1^{i+1\frown} 0$.

T will have a main path x divided into countably many levels. For every $B \in S$ our tree will have a countable number of elements $x_{b,s}$, one branching off from each level. The path $x_{b,s}$ will contain labels for all $i \in B$ and no others.

Define $Q_A = \{x \in Q : x \text{ has no labels for } i \notin A\}$.

Note $A \mapsto Q_A$ is a lattice homomorphism.

Condition (*)

The key to making $P_e =^* Q_A$ will be the following condition:

(*): For any $B \in S$ and P_e with A the parent of B, $|P_e \cap (Q_B - Q_A)| = \infty \Rightarrow |Q_B - P_e| < \infty$.

Lemma: (*) $\Rightarrow \forall P_e \subseteq Q \exists C \in L(Q_C =_* P_e)$. [1]

Proof: Fix $P_e \subseteq Q$ and let $C = \bigcup\{A \in S : |Q_A - P_e| < \infty\}$.

$|Q_C - P_e| < \infty$: $|Q_C - P_e| = |\bigcup\{Q_A - P_e\}| < \infty$.

$|P_e - Q_C| < \infty$: Assume otherwise and let $B \in S$ be of minimal cardinality with $|P_e \cap (Q_B - Q_C)| = \infty$. Let A be the parent of B. Then, by minimality, $|P_e \cap (Q_B - Q_A)| = \infty$ so by condition (*) $|Q_B - P_e| < \infty$, hence $Q_B \subseteq Q_C$ contradicting $|P_e \cap (Q_B - Q_C)| = \infty$.

Thus $P_e =^* Q_C$. $\square$

Construction

We build T in stages along with a computable function $n(s)$ such that by stage s we will have decided all paths of length $\leq n(s)$. This will ensure that T is computable.

During the construction we will have living paths which at each stage we will extend by a uniform amount. Any path that becomes dead will never live again thus allowing us to define $n(s)$ as the length of the living paths at the end of stage s.

We use the following symbols to track the living part of the tree:

As notation σ_k will be the kth level of the main path with $\sigma_k \succ \sigma_e$ for all $k > e$, and $x = \lim_k \lim_s \sigma_k^s$, the main path as a whole.

$\forall b \in \{0, \ldots, m\}$ and $k \in \omega$, we will have $\mu_{b,k}$ be a path with labels for b (corresponding to $B \in S$) and branching from σ_k, with limit $x_{b,k} = \lim_s \mu_{b,k}^s$.

Requirements

The following requirements will imply condition (*):

$R_{0,j,e}$ $(j > e)$: $x \in P_e \Rightarrow x_{0,j} \in P_e$.

$R_{b,j,e}$ $(b > 0, j > e)$:
$a \leq_* b$ and $x_{b,j} \in P_e \Rightarrow \forall k \geq j(x_{a,k} \in P_e)$.

Lemma: The requirements imply condition (*).

Proof: Fix B with parent A and suppose $|P_e \cap (Q_B - Q_A)| = \infty$. Fix $a \in B$ and b as the code for B. Now $x_{b,j} \in P_e$ for some $j \geq e$ so $R_{b,j,e}$ implies $x_{a,k} \in P_e$ for cofinitely many k. As a was chosen arbitrary from the finite set B, $|Q_B - P_C| < \infty$. □

We order requirements by level (the j's) and within the level by ancestor; a requirement is allowed to injure any of its ancestors.

We will work on the contrapositives of the requirements. Thus, if it looks like $x_{a,k} \notin P_e$ but $x_{b,j} \in P_e$ for suitable b and j, then we act to put $x_{b,j} \notin P_e$.

Action: Case All is OK

If no requirement needs action then we create branches off the end of the main path, one for each $A \in S$, and add another level to the main path. We can compute the longest string we add to these branches. We can then use this length to pad all living ends of the tree with 0's to the same length and define $n(s)$ as the length of the living part of the tree.

Action: Case $b = 0$

If $R_{0,j,e}$ requires action then we have $\sigma_j^s \in T_e$, $k > j$, and $\mu_{0,k}^s \notin T_e$. We are going to break off the tree above σ_j and reattach it at $\mu_{0,k}$. We then, as in the previous case, add on new branches to the end of the main path and extend the main path. Also as before we can compute the longest path and pad all living paths with 0's to that length.

Action: $b > 0$

If $R_{b,j,e}$ requires action then we have $\mu^s_{b,j} \in T_e$, $c \leq_* b$, $k > j$, and $\mu^s_{c,k} \notin T_e$.

Similar to the previous case we are going to break off the tree at σ_j and reattach it at σ_k, moving $\mu_{b,j}$ to $\mu_{c,k}$. Again we add on new branches to the end of the main path, extend the main path, calculate the new longest length, and pad to that length.

Verification

Note that acting on a requirement only injures requirements of lower priority; and that once a requirement acts it will remain valid until injured.

So, in a typical finite injury fashion, every requirement will only be injured a finite number of times and thus, after a finite number of stages, be satisfied and remains satisfied.

Note that x is a limit point and that $x_{a,k}$ is isolated for all a and k. Also note that x is not computable as then $\{x\}$ would be a subclass P_e, but then $x_{0,j} \in P_e$ for all $j \geq e$, a contradiction.

$\mathbf{Th}(\mathcal{L}(Q))$ is decidable.

Lemma: Let P be a countable Π^0_1 class with every computable member isolated. Then $\mathcal{L}(P)$ is isomorphic to a sublattice of $\mathcal{P}(\mathbb{N})$ closed under finite differences. [1]

Proof Sketch of Lemma: Let $A = \{\alpha_n : n \in \omega\}$ be the set of isolated points of P. We claim that $P_e \mapsto \{n : \alpha_n \in P_e \cap A\}$ is an isomorphism. Fix Q_1, Q_2 such that $Q_1 \cap A = Q_2 \cap A$, and $x \in Q_1$. By induction on the Cantor-Bendixson rank of x we show that any open neighborhood of x in Q_1 contains an element of A and thus of Q_2. As Q_2 is closed it follows that $x \in Q_2$. Similar for the reverse inclusion. So $Q_1 = Q_2$. Closed under finite differences follows as all α_n are computable. $\square$

Note that Q satisfies the hypothesis of the lemma. So by a theorem of Lachlan [3] $\mathrm{Th}(\mathcal{L}(Q)) \leq_m \mathrm{Th}(L)$. As L is a finite lattice it has decidable theory and thus $\mathrm{Th}(\mathcal{L}(Q))$ is decidable. $\square$

References

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[4] A. Nies. Effectively dense boolean algebras and their applications. *Trans. Amer. Math. Soc. 352*, pages 4989–5012, 2000.